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On the Solutions of Stochastic Initial-Value Problems in Continuum Mechanics

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This work deals with the analysis of a class of initial-value problems for non-linear hyperbolic equations with stochastic process coefficients and four time–space independent variables. A perturbation procedure is developed, leading to a sequence of stochastic, partial differential equations which are solved by application of the Adomian decomposition method. A pertinent application in mechanics is also considered, and quantitative results are presented and discussed. © 1985 Academic Press, Inc.

1. INTRODUCTION

A partial differential equation is often the abstract mathematical model of a real physical system such that the dependent variable, which defines the state of the system, is continuously distributed over the independent time–space variables. It is also well known [1, 2] that such equations arise in continuum mechanics and that each particular equation has to be characterized by the pertinent initial and boundary conditions, as well as by a set of parameters which may identify, for instance, the physical properties of the matter where the mathematically described phenomenon occurs.

For some physical systems, however, the coefficients of the considered equation and the initial and boundary conditions cannot be predicted in deterministic terms, so that the resulting equation is a random partial differential equation with stochastic process coefficients and random initial and boundary conditions. Some discussions on the conceivable

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stochasticities of physical systems in continuum mechanics have been proposed in Refs. [3, 4, 8], among others.

This paper deals with the analysis of a class of initial-value problems for nonlinearly perturbed hyperbolic equations, with particular attention to the ones arising as a mathematical model of some phenomena of wave propagation [5–7]. In particular, the mathematical statement of the problem is proposed in the second section, whereas the mathematical analysis is realized in the third and fourth sections. The said analysis is mainly based on the Adomian method [8], which has been recently proposed for partial differential equations [9, 10] as the natural extension of his decomposition method for random ordinary differential equations; see Ref. [11] or the review of Ref. [12]. The fifth section presents a particular application in mechanics with some quantitative results and a final discussion.

2. Mathematical Statement of the Problem

We consider dynamical systems which are mathematically described by the stochastic operator equation

$$\begin{aligned} &L_{t,x}^{(2)} u + R_{t,x}^{(2)} u = f(t, \mathbf{x}, u, L_{t,x}^{(1)} u, r(\mathbf{x}, t; \omega), \eta) \\ &u({}_x; t=0) = \psi_1(\mathbf{x}); u_t(\mathbf{x}; t=0) = \psi_2(\mathbf{x}) \end{aligned} \tag{1}$$

for which the following assumptions are made.

H.1. The dependent variable $u = u(\mathbf{x}, t; \omega)$ is a stochastic process defined as

$$u: \{u \in C^\infty([0, T]; \tilde{C}^\infty(\mathbb{R}^3)), (D_x \cdot I) \cdot \Omega \to D \subseteq R\},$$

where $\tilde{C}^\infty$ is the space of all differentiable functions which go to zero with their x_k-derivatives when $|x_k| \to \infty$, being

$$\mathbf{x} = \{x_k\} \in D_x \subseteq \mathbb{R}^3, k = 1, 2, 3; \qquad t \in I = [0, T]; \qquad \omega \in (\Omega, F, \mu).$$

H.2. In Eq. (1), $L_{t,x}^{(1)}$ and $L_{t,x}^{(2)}$ are linear, deterministic differential operators of the first and second order, respectively,

$$\begin{aligned} L_{t,x}^{(1)} &= L_t^{(1)} + L_x^{(1)} = \frac{\partial}{\partial t} + \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \\ L_{t,x}^{(2)} &= a \frac{\partial^2}{\partial t^2} + \sum_{k=1}^{3} b_k \frac{\partial^2}{\partial t\, \partial x_k} + \sum_{h,k=1}^{3} c_{hk} \frac{\partial^2}{\partial x_h\, \partial x_k} \end{aligned} \tag{2}$$

with a, b_k and c_{hk} known deterministic constants; whereas $R^{(2)}_{t,x}$ is a stochastic, linear differential operator given by

$$R^{(2)}_{t,x} = \alpha(\mathbf{x}, t; \omega)\frac{\partial^2}{\partial t^2} + \sum_{k=1}^{3} \beta_k(\mathbf{x}, t; \omega)\frac{\partial^2}{\partial t\, \partial x_k} + \sum_{h,k=1}^{3} \gamma_{hk}(\mathbf{x}, t; \omega)\frac{\partial^2}{\partial x_h\, \partial x_k} \tag{3}$$

with

$$\begin{aligned} \alpha(\mathbf{x}, t; \omega) &= \sum_p A_p(\omega)\, \phi_{1,p}(\mathbf{x}, t) \\ \beta_k(\mathbf{x}, t; \omega) &= \sum_q B^{(k)}_q(\omega)\, \phi^{(k)}_{2,q}(\mathbf{x}, t) \\ \gamma_{hk}(\mathbf{x}, t; \omega) &= \sum_r C^{(hk)}_r(\omega)\, \phi^{(hk)}_{3,r}(\mathbf{x}, t). \end{aligned} \tag{4}$$

A_p, $B^{(k)}_q$ and $C^{(hk)}_r$ are uncorrelated random variables with zero expected values, defined in the probability space (Ω, F, μ) with known probability densities which are constant with respect to $\mathbf{x}$ and t; whereas ϕ_i are known functions of the independent time–space variables. Note that the sums $a + \alpha(\mathbf{x}, t; \omega)$, $b_k + \beta_k(\mathbf{x}, t; \omega)$ and $c_{hk} + \gamma_{hk}(\mathbf{x}, t; \omega)$ are a simplified version of Karhunen–Loéve expansions [8] of the stochastic process coefficients which are involved in the considered equation.

H.3. It is assumed that each realization of $\alpha, \beta_k, \gamma_{hk}$ and the deterministic parameters a, b_k, c_{hk} satisfy the hyperbolicity conditions of Eq. (1) for $t \in I$ and $\mathbf{x} \in D_x$.

H.4. The nonlinear term f is a known function of $\mathbf{x}, t, u, L^{(1)}_{t,x}u$, depending on a deterministic parameter η and on a stochastic process $r(\mathbf{x}, t; \omega)$ whose expression may be given in the form

$$r(\mathbf{x}, t; \omega) = \sum_s D_s(\omega)\, \phi_{4,s}(\mathbf{x}, t) \tag{5}$$

with $D_s(\omega)$ a constant random variable and $\phi_{4,s}$ a known function of $\mathbf{x}$ and t. It is also assumed that for each realization of the stochastic processes $\alpha, \beta_k, \gamma_{hk}, r$, the function f is analytic with respect to its arguments:

$$f\colon \{f \in C^\infty([0, T]; \tilde{C}^\infty(\mathbb{R}^3)), (D_x \cdot I) \cdot \Omega \to B \subseteq \mathbb{R}\}.$$

H.5. The initial conditions are deterministic, with

$$\psi_{1,2}\colon \{\psi_{1,2} \in C^\infty([0, T]; \tilde{C}^\infty(\mathbb{R}^3)), D_x \to D_{1,2} \subseteq \mathbb{R}\}.$$

The problem defined by Eq. (1) and the above hypotheses is very general and, as known, is the mathematical model of a large class of physical systems in continuum mechanics which, in equilibrium or evolution conditions, are characterized by the following two kinds of randomness:

(i) randomness in the inner physical properties of the system, modelled by the processes α, β_k and γ_{hk};

(ii) randomness in the external actions, modelled by the process r.

In the mathematical description of a large number of real physical systems modelled as above, it is reasonable to assume that the nonlinear perturbations in the second member of Eq. (1) are small compared with the deterministic coefficients a, b_k, c_{hk}, so that the assumption: $\eta = o(1)$ may be quite realistic. Though this restriction is not necessary in our analysis and will later be removed, it is in conformity with much existing work and applicable to a wide class of phenomena. Consequently, in the next two sections, Eq. (1) will be studied by first considering η as a very small parameter, and developing a perturbation procedure leading to a sequence of stochastic linear problem to be solved by application of the Adomian method, and second, extending the method to nonlinear partial differential equations with η not small.

3. Analysis

In order to deal with the stochastic equation (1), let us rewrite it in the form

$$(a+\alpha)\frac{\partial^2 u}{\partial t^2} = f - \sum_{k=1}^{3} (b_k + \beta_k)\frac{\partial^2 u}{\partial t\,\partial x_k} - \sum_{h,k=1}^{3} (c_{hk} + \gamma_{hk})\frac{\partial^2 u}{\partial x_h\,\partial x_k}. \tag{6}$$

Introduction of the state vector,

$$\mathbf{u} = \{u_1 = u_t,\, u_2 = u_{x_1},\, u_3 = u_{x_2},\, u_4 = u_{x_3},\, u_5 = u\} \in \mathbb{R}^5, \tag{7}$$

leads to the following equivalent system of five stochastic partial differential equations:

$$\begin{aligned} &\frac{\partial u_j}{\partial t} = F_j(t, \mathbf{x}, u_i, \partial u_i/\partial x_k; \omega, \eta); \qquad i, j = 1,..., 5 \\ &u_j(\mathbf{x}, t=0) = u_{j,0}(\mathbf{x}) \end{aligned} \tag{8}$$

with

$$F_1=\left\{f-\sum_{k=1}^{3}\left[(b_k+\beta_k)\frac{\partial u_1}{\partial x_k}+\sum_{h=1}^{3}(c_{hk}+\gamma_{hk})\frac{\partial}{\partial x_k}u_{h+1}\right]\right\}\Big/(a+\alpha)$$
$$F_2=\frac{\partial u_1}{\partial x_1};\qquad F_3=\frac{\partial u_1}{\partial x_2};\qquad F_4=\frac{\partial u_1}{\partial x_3};\qquad F_5=u_1 \tag{9}$$

and

$$u_{1,0}(\mathbf{x})=\psi_2(\mathbf{x});\qquad u_{2,0}(\mathbf{x})=\frac{\partial\psi_1}{\partial x_1};\qquad u_{3,0}(\mathbf{x})=\frac{\partial\psi_1}{\partial x_2};$$
$$u_{4,0}(\mathbf{x})=\frac{\partial\psi_1}{\partial x_3};\qquad u_{5,0}(\mathbf{x})=\psi_1(\mathbf{x}). \tag{10}$$

As a consequence of the hypotheses H.1 and H.4, the considered functional space is a subspace of $L^2(\Omega)=L^2(\Omega, F, \mu)$ of the square-integrable functions of Ω. In such a subspace, the following inner product and norm are defined:

$$(f', f'')_\Omega=\int_\Omega f'\cdot f''\,dP=E(f', f'')$$
$$\|f\|_\Omega^2=\int_\Omega f\cdot f\,dP=E(f^2). \tag{11}$$

If P is the probability density induced by F, then the inner product and norm represent the expected values of their arguments. In the above-defined functional space, considering that both the function f and the initial conditions are analytic with respect to their arguments (see hypotheses H.4 and H.5), it follows from the Cauchy–Kowalewski theorem that the problem (8) has a unique local analytic solution, satisfying the conditions (10). Consequently, the "reduced" problem which is obtained by setting $\eta=0$ into Eq. (8) also has obviously a unique local solution $\mathbf{u}^{(0)}(\mathbf{x}, t, \omega; \eta=0)$, which in the following analysis will be assumed as known.

If $\mathscr{U}$ represents the functional space of the state vector, and $\mathscr{F}$ represents the space of the initial conditions and of the nonlinear term ($\mathscr{U}$, $\mathscr{F}$ normed linear spaces), the solution of Eq. (8) is continuously dependent on the initial data if, for given $f', f''\in\mathscr{F}$ and $u', u''\in\mathscr{U}$, there exists a constant K, independent of f' and f'', such that

$$\|u'-u''\|_{\mathscr{U}}\leqslant K\|f'-f''\|_{\mathscr{F}}. \tag{12}$$

Therefore if (12) holds we may conclude that the problem defined by Eq. (8) is a well-posed one under the hypotheses made in the preceding section.

Equation (8) can be rewritten as

$$\sum_{i=1}^{5}\left[\sum_{k=1}^{3} A_{ji}^{(k)}(\mathbf{x}, t; \omega)\frac{\partial u_i}{\partial x_k} + B_{ji}(\mathbf{x}, t; \omega)\frac{\partial u_i}{\partial t}\right]$$
$$= f_j(\mathbf{x}, t, u_i, \omega, \eta), \qquad i, j = 1,\ldots, 5 \tag{13}$$

with

$$f_1 = f; \qquad f_2 = f_3 = f_4 = 0; \qquad f_5 = u_1, \tag{14}$$

where the coefficients A_{ji}, B_{ji} are defined as

$$\begin{aligned}
A_{11}^{(k)} &= b_k + \beta_k(\mathbf{x}, t; \omega), \qquad k = 1, 2, 3\\
A_{1\ell}^{(k)} &= c_{\ell-1,k} + \gamma_{\ell-1,k}(\mathbf{x}, t; \omega), \qquad \ell = 2, 3, 4\\
A_{k+1,1}^{(k)} &= -1\\
B_{11} &= a + \alpha(\mathbf{x}, t; \omega)\\
B_{22} &= B_{33} = B_{44} = B_{55} = 1
\end{aligned} \tag{15}$$

and the other ones are null. In operator form we have

$$\sum_{i=1}^{5} (L_{ji} + R_{ji})\, u_i = f_j; \qquad u_j(\mathbf{x}, t=0) = u_{j,0}(\mathbf{x}) \tag{16}$$

with

$$\begin{aligned}
L_{11} &= \sum_{k=1}^{3} b_k \frac{\partial}{\partial x_k} + a\frac{\partial}{\partial t}; \qquad L_{\ell,1} = -\frac{\partial}{\partial x_{\ell-1}}, \qquad \ell = 2, 3, 4\\
L_{1,\ell} &= \sum_{k=1}^{3} c_{\ell-1,k}\frac{\partial}{\partial x_k}; \qquad L_{22} = L_{33} = L_{44} = L_{55} = \frac{\partial}{\partial t}
\end{aligned} \tag{17a}$$

and

$$\begin{aligned}
R_{11} &= \sum_{k=1}^{3} \beta_k(\mathbf{x}, t; \omega)\frac{\partial}{\partial x_k} + \alpha(\mathbf{x}, t; \omega)\frac{\partial}{\partial t}\\
R_{1,\ell} &= \sum_{k=1}^{3} \gamma_{\ell-1,k}(\mathbf{x}, t; \omega)\frac{\partial}{\partial x_k}, \qquad \ell = 2, 3, 4
\end{aligned} \tag{17b}$$

whereas the other differential operators are null.

Case of Small Nonlinearity

If $\eta \ll 1$, the solution of Eq. (16) can be sought as a power expansion in the small parameter η, in the form

$$u_j = \sum_{m=0}^{M} \eta^m u_j^{(m)}(\mathbf{x}, t; \omega) + O(\eta^{m+1})$$
$$u_j^{(0)}(\mathbf{x}, t=0; \omega) = u_{j0}; \quad u_j^{(m)}(\mathbf{x}, t=0; \omega) = 0, \quad m = 1, ..., M. \tag{18}$$

As a consequence of the hypothesis H.4, the functions f_j can be analogously developed [14] as

$$f_j \cong \sum_{m=0}^{M} \eta^m f_j^{(m)} \tag{19}$$

with

$$f_2^{(m)} = f_3^{(m)} = f_4^{(m)} = 0, \; f_5^{(m)} = u_1^{(m)}, \qquad m = 0, 1, ..., M$$
$$f_1^{(0)} = f(\mathbf{x}, t, u_i^{(0)}; r, \eta = 0)$$
$$f_1^{(m)} = \frac{1}{m!} \left(\frac{d^m f}{d\eta^m}\right)_{\eta=0} = \sum_{i=1}^{5} \left(\frac{df}{du_i}\right)_{\eta=0} u_i^{(m)} + \tilde{f}_1^{(m)}(\mathbf{x}, t, u_i^{(0)}, ..., u^{(m-1)}; r),$$
$$m = 1, ..., M,$$

where $\tilde{f}_1^{(m)}$ is a known function independent of the mth order term $u_i^{(m)}$ of the expansion (18). Substituting into Eq. (18) and equating the terms with equal power of η, the following sequence of initial-value problems is obtained:

$$\sum_{i=1}^{5} (L_{ji} + R_{ji})\, u_i^{(0)} = f_j(\mathbf{x}, t, u_i^{(0)}, r; \eta = 0), \quad u_j^{(0)}(\mathbf{x}, t = 0) = u_{j,0} \tag{20}$$

$$\sum_{i=1}^{5} (\tilde{L}_{ji} + \tilde{R}_{ji})\, u_i^{(m)} = \tilde{f}_j^{(m)}, \quad u_j^{(m)}(\mathbf{x}, t = 0) = 0 \quad \text{for } m = 1, 2, ..., M \tag{21}$$

with $\tilde{f}_2^{(m)} = \tilde{f}_3^{(m)} = \tilde{f}_4^{(m)} = \tilde{f}_5^{(m)} = 0$ and

$$\tilde{L}_{ji} = L_{ji} \text{ except: } \tilde{L}_{5,1} = -1$$
$$\tilde{R}_{ji} = R_{ji} \quad \text{for } j = 2, 3, 4, 5; \quad \tilde{R}_{1,i} = R_{1,i} - (\partial f/\partial u_i)_{\eta=0}, \tag{22}$$

where L_{ji} and R_{ji} are defined by Eq. (17). In particular, for $m = 1$, Eq. (21) is equivalent to the system

$$\sum_i (L_{1,i} + \tilde{R}_{1,i})\, u_i^{(1)} = (\partial f/\partial \eta)_{\eta=0}$$
$$\sum_i (\tilde{L}_{j,i} + R_{j,i})\, u_i^{(1)} = 0; \quad j = 2, ..., 5; \quad \mathbf{u}^{(1)}(\mathbf{x}, t = 0) = 0 \tag{21'}$$

and for $m=2$:

$$\sum_i (L_{1,i}+\tilde{R}_{1,i})\, u_i^{(2)} = \frac{1}{2}(\partial^2 f/\partial\eta^2)_{\eta=0}$$

$$+\frac{1}{2}\sum_i \left[\sum_{\ell=1}^{5} u_\ell^{(1)}(\partial^2 f/\partial u_i\, \partial u_\ell)_{\eta=0} + (\partial^2 f/\partial u_i\, \partial\eta)_{\eta=0} \right] u_i^{(1)} \quad (21'')$$

$$\sum_i (\tilde{L}_{j,i}+R_{j,i})\, u_i^{(2)} = 0; \quad j=2,\ldots,5; \quad \mathbf{u}^{(2)}(\mathbf{x}, t=0)=0.$$

The solution $\mathbf{u}^{(0)}=\{u_j^{(0)}\}$ of the nonlinear system (20) is known, since it is the solution of the reduced problem for $\eta=0$. Substitution of $u_j^{(0)}$ into Eq. (21′) leads to a linear, nonhomogeneous problem whose solution $\mathbf{u}^{(1)}=\{u_j^{(1)}\}$ yields the first term of the expansion (18). Repeating the procedure for $m=2, 3,\ldots, M$ leads to analogous systems of linear random differential equations whose solutions give the mth term of the approximated solution (18).

In studying a wide class of physical systems with small nonlinearities as, for instance, the one considered in the application of Section 5, the function f in Eq. (1) can be written in the form

$$f=\eta g(\mathbf{x}, t, u_i, r_1(\mathbf{x}, t, \omega)) + h(\mathbf{x}, t, r_2(\mathbf{x}, t, \omega)), \quad (23)$$

where the nonlinear term g is a known analytic function independent of η; h is a nonhomogeneous term, and r_1, r_2 are stochastic processes whose structure is defined by Eq. (5). For such physical systems, by setting

$$\begin{aligned}
&\mathbf{f}=\eta\mathbf{g}+\mathbf{h}\\
&\mathbf{g}=\{g_j\}=\{g_1=g,\ g_2=g_3=g_4=g_5=0\}\\
&\mathbf{h}=\mathbf{h}_1+\mathbf{h}_2 \qquad\qquad (23')\\
&\mathbf{h}_1=\{h_{1j}\}=\{h_{11}=h,\ h_{12}=h_{13}=h_{14}=h_{15}=0\}\\
&\mathbf{h}_2=\{h_{2j}\}=\{h_{21}=h_{22}=h_{23}=h_{24}=0,\ h_{25}=u_1\}
\end{aligned}$$

and developing the nonlinear function $\mathbf{g}$,

$$\mathbf{g} \cong \sum_{m=0}^{M} \eta^m \mathbf{g}^{(m)}, \quad (24)$$

with

$$\begin{aligned}
&g_1^{(0)}=g(\mathbf{x}, t, u_i^{(0)}, r_1); \quad g_1^{(m)}=(d^m g/d\eta^m)_{\eta=0}/m!, \quad m=1,\ldots, M\\
&g_2^{(m)}=g_3^{(m)}=g_4^{(m)}=g_5^{(m)}=0, \qquad m=0, 1,\ldots, M
\end{aligned} \quad (24')$$

then the sequence of initial-value problems for the unknown terms $u_j^{(m)}$ of the expansion (18) resulting from the above procedure is

$$\sum_{i=1}^{5} (\tilde{L}_{ji} + R_{ji})\, u_i^{(0)} = h_{1j}(\mathbf{x}, t, r_2(\mathbf{x}, t; \omega));\; u_j^{(0)}(\mathbf{x}, t=0) = u_{j,0} \tag{25}$$

$$\sum_{i=1}^{5} (\tilde{L}_{ji} + R_{ji})\, u_i^{(m)} = g_j^{(m-1)}(\mathbf{x}, t, u_i^{(0)}, ..., u_i^{(m-1)}; r_1);\; u_j^{(m)}(\mathbf{x}, t=0) = 0,$$

$$m = 1, ..., M. \tag{26}$$

This sequence replaces Eqs. (20), (21) if the condition (23) holds. We observe that in this case the same deterministic and stochastic operators $\tilde{L}_{ji}$ and R_{ji} are applied to both the zero-order and the mth-order unknown terms of the expansion (18).

4. Method of Solution

In this section, the Adomian decomposition method [9–11] for stochastic partial differential equations, suitably extended to vector state variables, is applied in order to solve the linear systems (21), (26) and successively the nonlinear operator equation (16), without the assumption of small nonlinearities. The solutions obtained are then utilized for determining the moments of the solution process.

Case of Small Nonlinearity

The linear systems (21), (26) are equivalent to the vector equation

$$[\mathscr{L}_{t,x}]\, \mathbf{u}^{(m)} = [L_{t,x}]\, \mathbf{u}^{(m)} + [R_{t,x}]\, \mathbf{u}^{(m)} = \chi; \qquad \mathbf{u}^{(m)}(\mathbf{x}, t=0) = \mathbf{0}, \tag{27}$$

where $[L_{t,x}]$ and $[R_{t,x}]$ are 5×5 operator matrices whose elements are given by Eqs. (22) and (17), respectively. Moreover, $\chi = \{\mathcal{J}_j^{(m)}\}$ in the case of Eq. (21) and $\chi = \{g_j^{(m-1)}\}$ in the case of Eq. (26). On the grounds of the hypotheses made in the previous sections, the inverse operator matrix $[\mathscr{L}_{t,x}]^{-1}$ exists. We now assume that the problem is also well posed when the stochastic operator $[R_{t,x}]$ vanishes. Since $[R_{t,x}]$ has null mean value (see the hypothesis H.2), then the inverse matrix $[L_{t,x}]^{-1}$ also exists. It follows that

$$[L_{t,x}]\, \mathbf{u}^{(m)} = \chi - [R_{t,x}]\, \mathbf{u}^{(m)} \tag{28}$$

$$\mathbf{u}^{(m)} = [\mathscr{L}_{t,x}]^{-1}\, \chi = [L_{t,x}]^{-1}\, \chi - [L_{t,x}]^{-1}\, [R_{t,x}]\, \mathbf{u}^{(m)}. \tag{29}$$

Application of the Adomian decomposition method yields the following operator equation for $[\mathscr{L}_{t,x}]^{-1}$ (see Ref. [11]):

$$[\mathscr{L}_{t,x}]^{-1} = \sum_{n=0}^{\infty} (-1)^n ([L_{t,x}]^{-1} [R_{t,x}])^n [L_{t,x}]^{-1}. \tag{30}$$

By considering Eq. (29), the above equation allows the calculation of $\mathbf{u}^{(m)}$ by performing the inversion of *only the deterministic* operator matrix, and supplies the result

$$\mathbf{u}^{(m)} = \sum_{n=0}^{\infty} (-1)^n ([L_{t,x}]^{-1} [R_{t,x}])^n [L_{t,x}]^{-1} \chi \tag{31}$$

whereas, by truncating at the n_mth term the series (31), the approximated values

$$\mathbf{w}^{(m)} = \sum_{n=0}^{n_m} (-1)^n ([L_{t,x}]^{-1} [R_{t,x}])^n [L_{t,x}]^{-1} \chi \tag{32}$$

are obtained. Under suitable convergence conditions (see Ref. [11]), which assure that

$$\lim_{n_m \to \infty} \mathbf{w}^{(m)} = \mathbf{u}^{(m)},$$

te approximated value $\mathbf{u}^*(\mathbf{x}, t; \omega)$ of the solution process for small non-linearities is therefore given by

$$\mathbf{u}^*(\mathbf{x}, t; \omega) = \mathbf{u}^{(0)}(\mathbf{x}, t; \omega) + \sum_{m=1}^{M} \eta^m \mathbf{w}^{(m)}(\mathbf{x}, t; \omega), \tag{33}$$

which converges to the solution process (18) of the considered problem for $n_m \to \infty$ and $M \to \infty$.

Case of Strong Nonlinearities

If η is not small, Eq. (16) must be solved instead of the linear sequence (27). In order to seek the solution of Eq. (16), the nonhomogeneous function $\mathbf{f} = \{f_j\}$ can be written in the form

$$\mathbf{f}(\mathbf{x}, t, u_i, \omega, \eta) = -[N]\, \mathbf{u} + \mathbf{Z}(\mathbf{x}, t; \omega), \tag{34}$$

where $[N]$ is a 5×5 matrix whose elements N_{ij} are nonlinear functions with both deterministic and stochastic components, and $\mathbf{Z}(\mathbf{x}, t; \omega)$ is a known function of the time–space variables, eventually depending on random parameters. Owing to the position (34), and as a consequence of the properties of the operators L_{ji}, R_{ji} defined by Eq. (17), Eq. (16) can be written in the form

$$[F]\,\mathbf{u} = ([L_{x_1}] + [L_{x_2}] + [L_{x_3}] + [L_t])\,\mathbf{u} + ([R_{x_1}] + [R_{x_2}] + [R_{x_3}] + [R_t])\,\mathbf{u} + [N]\,\mathbf{u} = \mathbf{Z}(\mathbf{x}, t; \omega), \qquad (35)$$

where $[L_{x_k}] = [L_{ji}^{(k)}]$, $[R_{x_k}] = [R_{ji}^{(k)}]$, with $k = 1, 2, 3$, are matrices of the first-order differential operators $\partial/\partial x_k$, and $[L_t] = [L_{ji}^{(t)}]$, $[R_t] = [R_{ji}^{(t)}]$ contain only the derivatives $\partial/\partial t$. Namely, the nonnull elements of the above matrices are

$$\begin{aligned}
&L_{1,1}^{(k)} = b_k(\partial/\partial x_k)\\
&L_{1,\ell}^{(k)} = c_{\ell-1,k}(\partial/\partial x_k), \qquad \ell = 2, 3, 4\\
&L_{1,1}^{(t)} = a(\partial/\partial t)\\
&L_{2,2}^{(t)} = L_{3,3}^{(t)} = L_{4,4}^{(t)} = L_{5,5}^{(t)} = \partial/\partial t\\
&R_{1,1}^{(k)} = \beta_k(\mathbf{x}, t; \omega)(\partial/\partial x_k)\\
&R_{1,\ell}^{(k)} = \gamma_{\ell-1,k}(\mathbf{x}, t; \omega)(\partial/\partial x_k), \qquad \ell = 2, 3, 4.
\end{aligned} \qquad (36)$$

The Eq. (35) has the same structure as the scalar operator equation, which is solved in Chapter X of Ref. [8] by application of the Adomian decomposition method: the operators do not contain mixed derivatives, and N_{ji}, $\mathbf{Z}$ are analytic with respect to their arguments, owing to the assumption made for the nonlinear term f of Eq. (1). Therefore, under sufficient convergence hypotheses and for each realization of the random parameters involved in the considered equation, the solution $\mathbf{u}$ of Eq. (35) can be obtained by decomposition of the operator matrix $[F]^{-1}$ and of $\mathbf{u}$, as specified in the above-quoted reference.

Equation (35) in operator form[1] is

$$F\mathbf{u} = [L_{x_1} + L_{x_2} + L_{x_3} + L_t]\,\mathbf{u} + [R_{x_1} + R_{x_2} + R_{x_3} + R_t]\,\mathbf{u} + N\mathbf{u} = \mathbf{Z}(\mathbf{x}, t; \omega),$$

i.e., $F\mathbf{u} = \mathbf{Z}$ in Adomian's form. The solution formally is $\mathbf{u} = F^{-1}\mathbf{Z}$ and is to be determined. More compactly we have

$$F\mathbf{u} = L\mathbf{u} + R\mathbf{u} + N\mathbf{u} = \mathbf{Z}, \qquad (37)$$

[1] As a result of very recent work, we can do this in matrix form also (see G. Adomian and R. Rach, On the solution of algebraic equations by the decomposition method, and Application of the decomposition method to inversion of matrices, *J. Math. Anal. Appl.*, in press.

where now L and R are multidimensional. Solving for $L\mathbf{u}$,

$$L\mathbf{u} = \mathbf{Z} - R\mathbf{u} - N\mathbf{u} \tag{38}$$

or

$$\begin{aligned} L_{x_1}\mathbf{u} &= \mathbf{Z} - R\mathbf{u} - N\mathbf{u} - L_{x_2}\mathbf{u} - L_{x_3}\mathbf{u} - L_t\mathbf{u} \\ L_{x_2}\mathbf{u} &= \mathbf{Z} - R\mathbf{u} - N\mathbf{u} - L_{x_1}\mathbf{u} - L_{x_3}\mathbf{u} - L_t\mathbf{u} \\ L_{x_3}\mathbf{u} &= \mathbf{Z} - R\mathbf{u} - N\mathbf{u} - L_{x_1}\mathbf{u} - L_{x_2}\mathbf{u} - L_t\mathbf{u} \\ L_t\mathbf{u} &= \mathbf{Z} - R\mathbf{u} - N\mathbf{u} - L_{x_1}\mathbf{u} - L_{x_2}\mathbf{u} - L_{x_3}\mathbf{u}. \end{aligned} \tag{38'}$$

Operating with inverses of $L_{x_1}, L_{x_2}, L_{x_3}, L_t$, which are easily obtained, we have

$$\begin{aligned} L_{x_1}^{-1}L_{x_1}\mathbf{u} &= L_{x_1}^{-1}\mathbf{Z} - L_{x_1}^{-1}R\mathbf{u} - L_{x_1}^{-1}N\mathbf{u} - L^{-1}L_{x_2}\mathbf{u} - L_{x_1}^{-1}L_{x_3}\mathbf{u} - L_{x_1}^{-1}L_t\mathbf{u} \\ L_{x_2}^{-1}L_{x_2}\mathbf{u} &= L_{x_2}^{-1}\mathbf{Z} - L_{x_2}^{-1}R\mathbf{u} - L_{x_2}^{-1}N\mathbf{u} - L_{x_2}^{-1}L_{x_1}\mathbf{u} - L_{x_2}^{-1}L_{x_3}\mathbf{u} - L_{x_2}^{-1}L_t\mathbf{u} \\ L_{x_3}^{-1}L_{x_3}\mathbf{u} &= L_{x_3}^{-1}\mathbf{Z} - L_{x_3}^{-1}R\mathbf{u} - L_{x_3}^{-1}N\mathbf{u} - L_{x_3}^{-1}L_{x_1}\mathbf{u} - L_{x_3}^{-1}L_{x_2}\mathbf{u} - L_{x_3}^{-1}L_t\mathbf{u} \\ L_t^{-1}L_t\mathbf{u} &= L_t^{-1}\mathbf{Z} - L_t^{-1}R\mathbf{u} - L_t^{-1}N\mathbf{u} - L_t^{-1}L_{x_1}\mathbf{u} - L_t^{-1}L_{x_2}\mathbf{u} - L_t^{-1}L_{x_3}\mathbf{u}. \end{aligned} \tag{39}$$

If L_{x_i} for $i = 1, 2, 3$ and L_t are first order, and suppressing ω for simplicity in writing and setting $\mathbf{x}$ for x_1, x_2, x_3,

$$L_{x_1}^{-1}L_{x_1}\mathbf{u} = \mathbf{u}(\mathbf{x}, t) - \mathbf{u}(-\infty, x_2, x_3, t) \tag{40a}$$

$$L_{x_2}^{-1}L_{x_2}\mathbf{u} = \mathbf{u}(\mathbf{x}, t) - \mathbf{u}(x_1, -\infty, x_3, t) \tag{40b}$$

$$L_{x_3}^{-1}L_{x_3}\mathbf{u} = \mathbf{u}(\mathbf{x}, t) - \mathbf{u}(x_1, x_2, -\infty, t) \tag{40c}$$

$$L_t^{-1}L_t\mathbf{u} = \mathbf{u}(\mathbf{x}, t) - \mathbf{u}(x_1, x_2, x_3, 0). \tag{40d}$$

In $C^\infty([0, T]; \tilde{C}^\infty(\mathbb{R}^3))$, the second terms of the right members in Eqs. (40a), (40b), (40c) are null, and the one in Eq. (40d) is known from the initial conditions. If the operators are higher than first order, additional term appear (Adomian [8]). Now we have

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t) &= L_{x_1}^{-1}\mathbf{Z} + \mathbf{u}(0, x_2, x_3, t) - L_{x_1}^{-1}R\mathbf{u} - L_{x_1}^{-1}\sum_{n=0}^{\infty}\mathbf{A}_n - L_{x_1}^{-1}L_{x_2}\mathbf{u} \\ &\quad - L_{x_1}^{-1}L_{x_3}\mathbf{u} - L_{x_1}^{-1}L_t\mathbf{u} \\ \mathbf{u}(\mathbf{x}, t) &= L_{x_2}^{-1}\mathbf{Z} + \mathbf{u}(x_1, 0, x_3, t) - L_{x_2}^{-1}R\mathbf{u} - L_{x_2}^{-1}\sum_{n=0}^{\infty}\mathbf{A}_n - L_{x_2}^{-1}L_{x_1}\mathbf{u} \\ &\quad - L_{x_2}^{-1}L_{x_3}\mathbf{u} - L_{x_2}^{-1}L_t\mathbf{u} \end{aligned} \tag{41}$$

$$\mathbf{u}(\mathbf{x}, t) = L_{x_3}^{-1}\mathbf{Z} + \mathbf{u}(x_1, x_2, 0, t) - L_{x_3}^{-1}R\mathbf{u} - L_{x_3}^{-1}\sum_{n=0}^{\infty}\mathbf{A}_n - L_{x_3}^{-1}L_{x_1}\mathbf{u}$$
$$- L_{x_3}^{-1}L_{x_2}\mathbf{u} - L_{x_3}^{-1}L_t\mathbf{u}$$

$$\mathbf{u}(\mathbf{x}, t) = L_t^{-1}\mathbf{Z} + \mathbf{u}(x_1, x_2, x_3, 0) - L_t^{-1}R\mathbf{u} - L_t^{-1}\sum_{n=0}^{\infty}\mathbf{A}_n - L_t^{-1}L_{x_1}\mathbf{u}$$
$$- L_t^{-1}L_{x_2}\mathbf{u} - L_t^{-1}L_{x_3}\mathbf{u},$$

where the nonlinear term $N\mathbf{u}$ is replaced by the appropriate Adomian polynomials defined for that $N\mathbf{u}$ and $\mathbf{u}$.

Now all four equations are added, and dividing by four (the dimensionality factor), we have

$$\begin{aligned}\mathbf{u} = {} & \tfrac{1}{4}\{L_{x_1}^{-1}\mathbf{Z} + L_{x_2}^{-1}\mathbf{Z} + L_{x_3}^{-1}\mathbf{Z} + L_t^{-1}\mathbf{Z} + \mathbf{u}(x_1, 0, x_3, t) + \mathbf{u}(x_1, x_2, 0, t) \\ & + \mathbf{u}(0, x_2, x_3, t) + \mathbf{u}(x_1, x_2, x_3, 0)\} - \tfrac{1}{4}\{L_{x_1}^{-1} + L_{x_2}^{-1} + L_{x_3}^{-1} + L_t^{-1}\}\, R\mathbf{u} \\ & - \tfrac{1}{4}\left\{(L_{x_1}^{-1} + L_{x_2}^{-1} + L_{x_3}^{-1} + L_t^{-1})\sum_{n=0}^{\infty}\mathbf{A}_n\right\} \\ & - \tfrac{1}{4}\{L_{x_1}^{-1}L_{x_2} + L_{x_1}^{-1}L_{x_3} + L_{x_1}^{-1}L_t + L_{x_2}^{-1}L_{x_1} + L_{x_2}^{-1}L_{x_3} + L_{x_2}^{-1}L_t \\ & + L_{x_3}^{-1}L_{x_1} + L_{x_3}^{-1}L_{x_2} + L_{x_3}^{-1}L_t + L_t^{-1}L_{x_1} + L_t^{-1}L_{x_2} + L_t^{-1}L_{x_3}\}\,\mathbf{u}.\end{aligned} \tag{42}$$

The term involving $\mathbf{Z}$ and the initial condition terms are known and the entire first term will be called $\mathbf{u}_0$ in the Adomian decomposition.

Thus for simplicity in writing if $L^{-1} = L_{x_1}^{-1} + L_{x_2}^{-1} + L_{x_3}^{-1} + L_t^{-1}$ and $R = R_{x_1} + R_{x_2} + R_{x_3} + R_t$,

$$\mathbf{u} = \mathbf{u}_0 - \tfrac{1}{4}L^{-1}\, R\mathbf{u} - \tfrac{1}{4}L^{-1}\sum_{n=0}^{\infty}\mathbf{A}_n - \tfrac{1}{4}\mathfrak{M}\mathbf{u}, \tag{43}$$

where $\mathfrak{M}$ has been used as a symbol for the operator products in the last term. As discussed in Refs. [9–11], we now have

$$\begin{aligned}\mathbf{u}_1 &= -\tfrac{1}{4}L^{-1}R\mathbf{u}_0 - \tfrac{1}{4}L^{-1}\mathbf{A}_0 - \tfrac{1}{4}\mathfrak{M}\mathbf{u}_0 \\ \mathbf{u}_2 &= -\tfrac{1}{4}L^{-1}R\mathbf{u}_1 - \tfrac{1}{4}L^{-1}\mathbf{A}_1 - \tfrac{1}{4}\mathfrak{M}\mathbf{u}_1 \\ &\vdots \\ \mathbf{u}_{n+1} &= -\tfrac{1}{4}\{L^{-1}R\mathbf{u}_n + L^{-1}\mathbf{A}_n + \mathfrak{M}\mathbf{u}_n\}\end{aligned} \tag{44}$$

and the solution we seek is $\mathbf{u} = \sum_{n=0}^{\infty}\mathbf{u}_n$, where the appropriate $\mathbf{A}_n$ are inserted and the operators are all known. The convergence of such solutions has been studied by Adomian [15]. Convergence is rapid and the terms are computable with very important advantages over usual methods since strong nonlinearities can be handled and less general results arise as special cases. For example, if $N\mathbf{u}$ is simply $\mathbf{u}$ rather than a nonlinear function of $\mathbf{u}$, the general term $\mathbf{A}_n$ is simply $\mathbf{u}_n$.

In order to calculate the moments of the solution process $\mathbf{u}$, it is convenient to introduce the random vector

$$\boldsymbol{\rho}(\omega) = \{A_p, B_q, C_r, D_s\} \in D_\rho, \tag{45}$$

whose components are defined by the set of random variables appearing in Eqs. (4) and (5). Therefore, the random vector $\boldsymbol{\rho}$ is defined in the probability space (Ω, F, μ) and involves a known probability density $P_\rho(\boldsymbol{\rho})$ which is constant with respect to t and $\mathbf{x}$. The moments of order p of the solution process are obtained by integration over D_ρ:

$$E\{\mathbf{u}^p\} = \int_{D_\rho} \mathbf{u}^p(\mathbf{x}, t; \boldsymbol{\rho})\, P_\rho(\boldsymbol{\rho})\, d\boldsymbol{\rho}. \tag{46}$$

In particular, for small nonlinearities, the *mean value* $\mathbf{m}^*(\mathbf{x}, t)$ of the above approximated solution is given by

$$\mathbf{m}^*(\mathbf{x}, t) = E\{\mathbf{u}^*\} = E\{\mathbf{u}^{(0)}\} + \sum_{m=1}^{M} \eta^m E\{\mathbf{w}^{(m)}\} \tag{47}$$

with

$$E\{\mathbf{w}^{(m)}\} = \int_{D_\rho} \mathbf{w}^{(m)}(\mathbf{x}, t; \boldsymbol{\rho})\, P_\rho(\boldsymbol{\rho})\, d\boldsymbol{\rho}, \tag{48}$$

where $\mathbf{w}^{(m)}(\mathbf{x}, t; \boldsymbol{\rho})$ is given by Eq. (32). The *correlation functions* $\Gamma^*_{ij}(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2)$ of two components of $\mathbf{u}^*$ are

$$\begin{aligned}
\Gamma^*_{ij}(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2) &= E\{u^*_i(\mathbf{x}_1, t_1)\, u^*_j(\mathbf{x}_2, t_2)\} \\
&= E\{u^{(0)}_i(\mathbf{x}_1, t_1)\, u^{(0)}_j(\mathbf{x}_2, t_2)\} + \sum_{m=1}^{M} \eta^m(\Gamma^{(0,m)}_{ij} + \Gamma^{(m,0)}_{ij}) \\
&\quad + \sum_{m=1}^{M} \sum_{\nu=1}^{M} \eta^{(m+\nu)} \Gamma^{(m,\nu)}_{ij},
\end{aligned} \tag{49}$$

where Γ_{ij} are the following cross-correlation functions of the terms $u^{(0)}_i$ and $w^{(m)}_i$:

$$\begin{aligned}
\Gamma^{(0,m)}_{ij}(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2) &= \int_{D_\rho} u^{(0)}_i(\mathbf{x}_1, t_1; \boldsymbol{\rho})\, w^{(m)}_j(\mathbf{x}_2, t_2; \boldsymbol{\rho})\, P_\rho(\boldsymbol{\rho})\, d\boldsymbol{\rho} \\
\Gamma^{(m,0)}_{ij}(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2) &= \int_{D_\rho} w^{(m)}_i(\mathbf{x}_1, t_1; \boldsymbol{\rho})\, u^{(0)}_j(\mathbf{x}_2, t_2; \boldsymbol{\rho})\, P_\rho(\boldsymbol{\rho})\, d\boldsymbol{\rho} \\
\Gamma^{(m,\nu)}_{ij}(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2) &= \int_{D_\rho} w^{(m)}_i(\mathbf{x}_1, t_1; \boldsymbol{\rho})\, w^{(\nu)}_j(\mathbf{x}_2, t_2; \boldsymbol{\rho})\, P_\rho(\boldsymbol{\rho})\, d\boldsymbol{\rho}.
\end{aligned} \tag{50}$$

Moreover, the *variance* of u_j^* at time t and at the point $\mathbf{x} = \{x_k\}$ can be obtained as follows:

$$V_j(\mathbf{x}, t) = \sigma^2\{u_j^*\} = \sigma^2\{u^{(0)}\} + 2 \sum_{m=1}^{M} \eta^m C_{jj}^{(0,m)}(\mathbf{x}, t; \mathbf{x}, t)$$

$$+ \sum_{m=1}^{M} \sum_{v=1}^{M} \eta^{(m+v)} C_{jj}^{(m,v)}(\mathbf{x}, t; \mathbf{x}, t), \tag{51}$$

where C_{jj} are the covariance functions

$$\begin{aligned} C_{jj}^{(0,m)}(\mathbf{x}, t; \mathbf{x}, t) &= \Gamma_{jj}^{(0,m)}(\mathbf{x}, t; \mathbf{x}, t) - E\{u_j^{(0)}\}\, E\{w_j^{(m)}\} \\ C_{jj}^{(m,v)}(\mathbf{x}, t; \mathbf{x}, t) &= \Gamma_{jj}^{(m,v)}(\mathbf{x}, t; \mathbf{x}, t) - E\{w_j^{(m)}\}\, E\{w_j^{(v)}\}, \end{aligned} \tag{52}$$

which can be calculated by making use of Eqs. (48) and (50).

5. Application and Conclusions

As an example of the method of solution developed in the case of small nonlinearities, here applied in the functional space,

$$(u, f, \psi_{1,2}) \in C^\infty, \qquad (D_x \cdot I) \cdot \Omega \to D \subseteq \mathbb{R},$$

where the problem is also well posed, consider the nonlinear wave problem

$$u_{tt} - c^2(\omega)\, u_{xx} = -\varepsilon b(\omega, t)\, u_t^2 \tag{53}$$

for the scalar variable $u(x, t; \omega)$ with (deterministic) initial conditions

$$u(x, 0) = \psi_1(x) = \sin x; \qquad u_t(x, 0) = \psi_2(x) = 0. \tag{54}$$

Assume that ε is a small deterministic parameter, whereas the coefficients $c(\omega)$ and $b(\omega, t)$ are known in probabilistic terms as follows:

$$\begin{aligned} c(\omega) &= c_0 + a c_1(\omega),\ c_1(\omega)\colon \Omega \to C \subset \mathbb{R} \\ b(\omega, t) &= \gamma b_1(\omega)\, \phi(t),\ b_1(\omega)\colon \Omega \to B \subset \mathbb{R}. \end{aligned} \tag{55}$$

$c_1(\omega)$ and $b_1(\omega)$ are known random variables with means zero and $\langle b_1 \rangle$ and probability densities $P_c(c_1)$, $P_b(b_1)$, respectively; $\phi(t)$ is a known deterministic function of time and c_0, a, γ are deterministic constants.

The system of first-order partial differential equations equivalent to Eq. (53) is supplied by Eqs. (8)–(10) of Section 3, which, in the considered unidimensional problem with respect to the spatial coordinates, are rewritten as

$$\frac{\partial u_j}{\partial t}=F_j\left(x,t,u_i,\frac{\partial u_i}{\partial x};\omega,\varepsilon\right),\quad u_j(x,0)=u_{j,0}(x);\quad j,i=1,2,3 \tag{56}$$

with

$$\begin{aligned}&\mathbf{u}(x,t;\omega)=\{u_1=u_t,u_2=u_x,u_3=u\}\\ &F_1=(c_0+ac_1(\omega))^2\,(\partial u_2/\partial x)-\varepsilon\gamma b_1(\omega)\,\phi(t)\,u_1^2;\\ &F_2=\partial u_1/\partial x;\ F_3=u_1\end{aligned} \tag{57}$$

and

$$u_{1,0}=0,\qquad u_{2,0}(x)=\cos x,\qquad u_{3,0}(x)=\sin x. \tag{58}$$

In operator form,

$$\sum_{i=1}^{3}(L_{ji}+R_{ji})\,u_i=f_j,\qquad u_j(x,0)=u_{j,0}(x), \tag{59}$$

being

$$\begin{aligned}&L_{11}=\partial/\partial t,\quad L_{12}=-c_0^2(\partial/\partial x),\quad L_{21}=-\partial/\partial x,\quad L_{22}=L_{33}=\partial/\partial t,\\ &\quad R_{12}=-(2ac_0c_1(\omega)+a^2c_1^2(\omega))(\partial/\partial x)\\ &L_{13}=L_{23}=L_{31}=L_{32}=R_{11}=R_{21}=R_{31}=R_{22}=R_{23}=R_{33}=R_{13}=R_{32}=0\end{aligned} \tag{60}$$

and

$$f_1=-\varepsilon\gamma b_1(\omega)\,\phi(t)\,u_1^2,\qquad f_2=0,\qquad f_3=u_1. \tag{61}$$

The ε^2-order approximated solution of Eq. (59) is now determined by application of the perturbation technique developed above:

$$\mathbf{u}^*(x,t;\omega)=\mathbf{u}^{(0)}(x,t;\omega)+\varepsilon\mathbf{u}^{(1)}(x,t;\omega)+\varepsilon^2\mathbf{u}^{(2)}(x,t;\omega). \tag{62}$$

The unknown terms of the truncated expansion are obtained by solving the linear initial-value problems

$$[\mathscr{L}_{t,x}]\,\mathbf{u}^{(0)}=\mathbf{0}\qquad \mathbf{u}^{(0)}(x,0)=\{u_{j,0}\} \tag{63a}$$

$$[\mathscr{L}_{t,x}]\,\mathbf{u}^{(1)}=\mathbf{g}^{(0)}\qquad \mathbf{u}^{(1)}(x,0)=\mathbf{0} \tag{63b}$$

$$[\mathscr{L}_{t,x}]\,\mathbf{u}^{(2)}=\mathbf{g}^{(1)}\qquad \mathbf{u}^{(2)}(x,0)=\mathbf{0}, \tag{63c}$$

where $[\mathscr{L}_{t,x}]$ is the operator matrix[2]

$$[\mathscr{L}_{t,x}] = [L_{t,x}] + [R_{t,x}]; \qquad [L_{t,x}] = [\tilde{L}_{ji}], \qquad [R_{t,x}] = [R_{ji}], \quad (64)$$

being $\tilde{L}_{ji} = L_{ji}$ with the exception of $L_{31} = -1$; moreover, in Eqs. (63) the components of the nonhomogeneous terms $\mathbf{g}^{(0)}$, $\mathbf{g}^{(1)}$ are, respectively,

$$\begin{aligned} g_1^{(0)}(x, t; \omega) &= -\gamma b_1(\omega)\, \phi(t)[u_1^{(0)}]^2, \qquad & g_2^{(0)} = g_3^{(0)} = 0 \\ g_1^{(1)}(x, t; \omega) &= -2\gamma b_1(\omega)\, \phi(t)\, u_1^{(0)} u_1^{(1)}, \qquad & g_2^{(1)} = g_3^{(1)} = 0. \end{aligned} \quad (65)$$

Equation (63a) defines the well-known linear wave problem whose classical solution satisfying the conditions (58) is

$$\begin{aligned} u_1^{(0)}(x, t; c(\omega)) &= -c(\omega) \sin x \sin\{c(\omega)\, t\} \\ u_2^{(0)}(x, t; c(\omega)) &= \cos x \cos\{c(\omega)\, t\} \\ u_3^{(0)}(x, t; c(\omega)) &= \sin x \cos\{c(\omega)\, t\}. \end{aligned} \quad (66)$$

In order to solve Eqs. (63b), (63c), the operator matrix $[\mathscr{L}_{t,x}]$, which has both a deterministic and a random part, is inverted with the result

$$[\mathscr{L}_{t,x}]^{-1} = [l_{ji}] = \begin{bmatrix} -c \sin x \sin(ct) + c(\partial I_0/\partial t)/2 & 0 & 0 \\ \cos x \cos(ct) + c(\partial I_0/\partial x)/2 & 0 & 0 \\ \sin x \cos(ct) + cI_0/2 & 0 & 0 \end{bmatrix} \quad (67)$$

with

$$I_0(x, t; \omega) = \int_0^t d\tau \int_{x - c(t-\tau)}^{x + c(t-\tau)} (\cdot)\, d\xi. \quad (68)$$

The solution for $\mathbf{u}^{(1)}$ is therefore

$$\mathbf{u}^{(1)} = \{u_j^{(1)}\}, \quad u_j^{(1)} = -\gamma b_1(\omega)\, l_{j1}[(u_1^{(0)})^2\, \phi(t)], \quad j = 1, 2, 3 \quad (69)$$

and the second-order term $\mathbf{u}^{(2)}$ in the solution (62) is given by

$$\mathbf{u}^{(2)} = \{u_j^{(2)}\}, \quad u_j^{(2)} = 2\gamma^2 b_1^2(\omega)\, l_{j1}[u_1^{(0)} \phi(t)\, l_{11}[(u_1^{(0)})^2\, \phi(t)]]. \quad (70)$$

[2] The earlier procedure for the case of strong nonlinearities did not require truncations and is much more accurate for that case, and in the case where nonlinearities become small, reduces to the perturbation results, which also result from truncations [8].

Detailed calculations for the component $u_3 = u$ of the perturbation terms given by Eqs. (69) and (70), which have been made by assuming $\phi(t) = 1 - \sin(\nu t + \theta)$, lead to the final result

$$
\begin{aligned}
&u^{(1)}(x, t; \omega) \\
&\quad = -\gamma b_1(\omega)\{\sin x \cos\{c(\omega)\, t\} + c^3(\omega)\, H_0(x, t; c(\omega))/2\} \\
&u^{(2)}(x, t; \omega) \\
&\quad = 2\gamma^2 b_1^2(\omega)\{\sin x \cos\{c(\omega)\, t\} + c^4(\omega)\, H_1(x, t; c(\omega))/2\},
\end{aligned}
\tag{71}
$$

where

$$
\begin{aligned}
H_0(x, t; c) &= c\{I_1(t; c) - I_2(t; c)\} - \tfrac{1}{2}\cos(2x)\{I_3(t; c) - I_4(t; c)\} \\
I_1(t; c) &= \int_0^t \sin^2(c\tau)(t-\tau)\, d\tau \\
I_2(t; c) &= \int_0^t \sin^2(c\tau)\sin(\nu\tau + \theta)(t-\tau)\, d\tau \\
I_3(t; c) &= \int_0^t \sin^2(c\tau)\sin\{2c(t-\tau)\}\, d\tau \\
I_4(t; c) &= \int_0^t \sin^2(ct)\sin(\nu\tau + \theta)\sin\{2c(t-\tau)\}\, d\tau \\
H_1(x, t; c) &= \frac{1}{4}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} \sin^2\xi\, \sin^2(c\tau)\{1 - \sin(\nu\tau + \theta)\} \\
&\quad \times \{2c[I_1'(\tau; c) - I_2'(\tau; c)] + \cos(2\xi)[I_3'(\tau; c) - I_4'(\tau; c)] \\
&\quad - \sin\xi\, \sin(c\tau)\}\, d\xi,
\end{aligned}
\tag{72}
$$

being $I_k'(t; c) = dI_k(t; c)/dt$, $k = 1, \ldots, 4$.

First- and second-order moments of the above approximated solution can now be determined by making use of the obtained results. The expected value of u^* is

$$
\begin{aligned}
E\{u^*\} = E\{u^{(0)}\} &- \varepsilon\gamma\langle b_1\rangle[E\{u^{(0)}\} + E\{c^3 H_0\}/2] \\
&+ 2\varepsilon^2\gamma^2\langle b_1^2\rangle[E\{u^{(0)}\} + E\{c^4 H_1\}/2],
\end{aligned}
\tag{73}
$$

where $\langle b_1\rangle$, $\langle b_1^2\rangle$ are the known first- and second-order moments of $b_1(\omega)$, and

$$E\{u^{(0)}\} = \int_C \sin x \cos[(c_0 + ac_1)t]\, P_c(c_1)\, dc_1$$

$$E\{c^3 H_0\} = \int_C (c_0 + ac_1)^3 H_0(x, t; c_0 + ac_1)\, P_c(c_1)\, dc_1 \tag{74}$$

$$E\{c^4 H_1\} = \int_C (c_0 + ac_1)^4 H_1(x, t; c_0 + ac_1)\, P_c(c_1)\, dc_1.$$

Finally, the variance of u^*, correct to the second-order terms with respect to the small parameter ε, can be calculated as

$$\sigma^2\{u^*\} = \sigma^2\{u^{(0)}\} + 2\varepsilon[E\{u^{(0)}u^{(1)}\} - E\{u^{(0)}\}\, E\{u^{(1)}\}] \\ + \varepsilon^2[\sigma^2\{u^{(1)}\} + 2[E\{u^{(0)}u^{(2)}\} - E\{u^{(0)}\}\, E\{u^{(2)}\}]], \tag{75}$$

where

$$E\{u^{(0)}u^{(1)}\} = \int_{B \cdot C} u^{(0)}(x, t; c_1)\, u^{(1)}(x, t; c_1, b_1)\, P_b(b_1)\, P_c(c_1)\, db_1\, dc_1 \\ = -\gamma\langle b_1\rangle[E\{u^{(0)2}\} + \sin x\, E\{c^3 H_0 \cos(ct)\}/2] \tag{76}$$

$$E\{u^{(0)}u^{(2)}\} = \int_{B \cdot C} u^{(0)}(x, t; c_1)\, u^{(2)}(x, t; c_1, b_1)\, P_b(b_1)\, P_c(c_1)\, db_1\, dc_1 \\ = 2\gamma^2\langle b_1^2\rangle[E\{u^{(0)2}\} + \sin x\, E\{c^4 H_1 \cos(ct)\}/2] \tag{77}$$

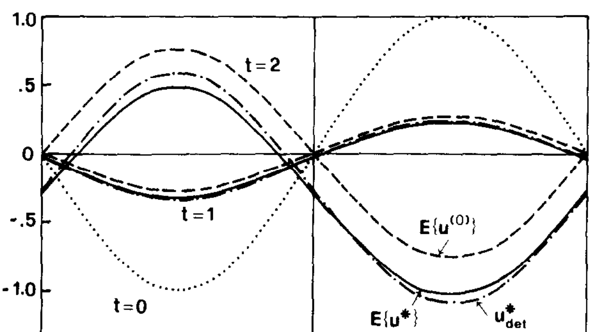

FIG. 1. Expected values of the solution process of Eq. (53) and comparisons, for $\gamma = 0.1$, $\langle b_1\rangle = 1$, $c_0 = 5$, $a = 1$, $v = 3$, $\theta = \pi/3$, $\varepsilon = 0.01$.

with

$$E\{c^3 H_0 \cos(ct)\} = \int_C (c_0 + ac_1)^3 \cos[(c_0 + ac_1)\, t]\, H_0(x, t; c_0 + ac_1)\, P_c(c_1)\, dc_1$$
$$E\{c^4 H_1 \cos(ct)\} = \int_C (c_0 + ac_1)^4 \cos[(c_0 + ac_1)\, t]\, H_1(x, t; c_0 + ac_1)\, P_c(c_1)\, dc_1. \tag{78}$$

Figures 1 and 2 show at $t = 1$ and $t = 2$ the expected values and variances of the solution process u^*, as they are deduced from the above equations in the ε-order approximation, and by assuming that the random variable $c_1(\omega)$ is beta-distributed within the values $(-\frac{1}{2}, \frac{1}{2})$ with probability density $P_c(c_1) = 6(\frac{1}{4} - c_1^2)$. The comparison of these results with the corresponding moments of the zero-order approximation $u^{(0)}$ shows the effects induced by the nonlinearity to the dynamical state of the system. Figure 1 also shows the deterministic solution u^*_{det}, which should be obtained in the simplified assumption that the coefficients $c(\omega)$ and $b_1(\omega)$ in Eq. (53) have deterministic values equal to their means: $\langle c \rangle = c_0$ and $\langle b_1 \rangle$, respectively. The comparison between the values of u^*_{det} and $E\{u^*\}$, together with the analysis of the variance $\sigma^2\{u^*\}$, provides knowledge of the statistical properties of the solution and allows us to conclude that the probabilistic treatment of the wave problem is necessary, in order to obtain accurate results for the behaviour of the solution process.

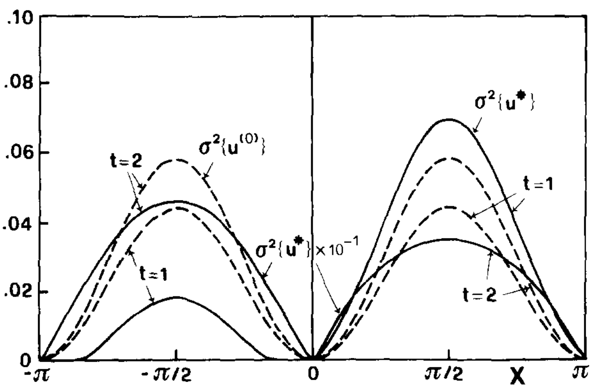

FIG. 2. Variances of the solution process of Eq. (53).

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